

# Numerical shadow and geometry of quantum states

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**Abstract.** The totality of normalised density matrices of order  $N$  forms a convex set  $\mathcal{Q}_N$  in  $\mathbb{R}^{N^2-1}$ . Working with the flat geometry induced by the Hilbert–Schmidt distance we consider images of orthogonal projections of  $\mathcal{Q}_N$  onto a two-plane and show that they are similar to the numerical ranges of matrices of order  $N$ . For a matrix  $A$  of a order  $N$  one defines its numerical shadow as a probability distribution supported on its numerical range  $W(A)$ , induced by the unitarily invariant Fubini–Study measure on the complex projective manifold  $\mathbb{C}P^{N-1}$ . We define generalized, mixed-states shadows of  $A$  and demonstrate their usefulness to analyse the structure of the set of quantum states and unitary dynamics therein.

PACS numbers: 02.10.Yn, 02.30.Tb, 03.67.-a

April 14, 2011

## 1. Introduction

Investigation of the geometry of the set of quantum states remains a subject of current scientific interests in view of possible applications in the theory of quantum information processing. The set  $\Omega_N$  of pure quantum states belonging to a  $N$  dimensional complex Hilbert space  $\mathcal{H}_N$  is known to be equivalent to the complex projective space,  $\Omega_N = \mathbb{C}P^{N-1}$ , of  $2N - 2$  real dimensions. However, as this set is embedded into the  $N^2 - 1$  dimensional set  $\mathcal{Q}_N$  of density matrices of size  $N$  by a non-linear constraint,  $\rho = \rho^2$ , the geometric structure of the set of mixed quantum states is rather involved [1, 2]. The only simple case corresponds to the one-qubit system,  $N = 2$ .

The set  $\Omega_2$  of  $N = 2$  pure states forms the *Bloch sphere*,  $\mathbb{C}P^1 = S^2$ , with respect to the standard Hilbert-Schmidt metric. The 3-disk inside the sphere, often called the *Bloch ball*, represents the set  $\mathcal{Q}_2$  of one-qubit mixed states. In this simple case any projection of this set onto a plane forms an ellipse, which can be degenerated to an interval. In the case of  $N = 3$  the 8 dimensional set  $\mathcal{Q}_3$  of one-qutrit mixed states is neither a polytope nor an ellipsoid [3, 4, 5], and the set  $\Omega_3 = \mathbb{C}P^2$  of its extremal states is connected and has four real dimensions.

Due to the high dimensionality of the problem our understanding of the geometry of the set  $\mathcal{Q}_N$  of mixed states is still rather limited. This set forms a convex body which contains an in-ball of radius  $r_N = \sqrt{1/N(N-1)}$  and can be inscribed into an out-sphere of radius  $R_N = (N-1)r_N = \sqrt{(N-1)/N}$  [2]. Some information on the subject can be gained by studying the 2-dimensional cross-sections of  $\mathcal{Q}_N$  as demonstrated in [6, 7, 8] for  $N = 3$  and  $N = 4$ . Another option is to investigate projections of this set into a plane – such an approach was advocated for  $N = 3$  in [9]. As the set  $\mathcal{Q}_N$  of quantum states is convex, also its cross-sections and projections inherit convexity.

In this work we study the general structure of a two-dimensional projection of the set  $\mathcal{Q}_N$  of mixed states. A bridge between the geometry of the set of quantum states and the notion of numerical range used in operator theory is established. For any operator  $A$ , acting on the complex Hilbert space  $\mathcal{H}_N$ , one defines its *numerical range* [10, 11] (also called *field of values*) as a subset of the complex plane which contains expectation values of  $A$  among arbitrary normalized pure states,

$$W(A) = \{z : z = \langle \psi | A | \psi \rangle, |\psi\rangle \in \mathcal{H}_N, \langle \psi | \psi \rangle = 1\}. \quad (1)$$

We analyse the set of orthogonal projections of the set  $\mathcal{Q}_N$  onto a 2-plane and prove that it is equivalent to the set of all possible numerical ranges of complex matrices of order  $N$ . Numerical ranges of normal matrices of size  $N$  correspond to orthogonal projections of the set  $\mathcal{C}_N$  of classical states – the  $(N-1)$ -dimensional simplex  $\Delta_{N-1} \subset \mathbb{R}^{N-1}$ .

Further information on the structure of the set of quantum states of a size  $N$  can be obtained by studying the *numerical shadow* [12, 13, 14] of various matrices of order  $N$ . For any operator  $A$  acting on  $\mathcal{H}_N$  one defines a probability distribution  $P_A(z)$  on the complex plane, supported in the numerical range  $W(A)$ ,

$$P_A(z) := \int_{\Omega_N} d\mu(\psi) \delta(z - \langle \psi | A | \psi \rangle). \quad (2)$$

Here  $\mu(\psi)$  denotes the unique unitarily invariant (Fubini-Study) measure on the set  $\Omega_N$  of  $N$ -dimensional pure quantum states. In other words the shadow  $P$  of matrix  $A$  at a given point  $z$  characterizes the likelihood that the expectation value of  $A$  among a random pure state is equal to  $z$ .

The distribution  $P_A(z)$  is naturally associated a given matrix  $A$ , and some of its properties were described in [13]. In this work we advocate a complementary approach and show that

investigating the shadows of several different complex matrices  $A$  of a fixed size  $N$  contributes to our understanding of the structure of the entire set  $\mathcal{Q}_N$  of quantum states. In a sense, the choice of a matrix  $A$  corresponds to the selection of the plane, onto which the set of quantum states is projected.

This paper is organized as follows. In section 2 we fix the notation and introduce necessary concepts. A link between 2-dimensional projections of the set of quantum states of a given size  $N$  and the set of possible numerical ranges of matrices of order  $N$  is presented in section 3. In section 4 we analyze different classes of numerical shadows of matrices of small order  $N = 2, 3, 4$ . Unitary dynamics of a pure quantum state in the background of numerical shadow is presented in section 5. Section 6 is devoted to the mixed-states numerical shadow, which corresponds to a projection of the full set  $\mathcal{Q}_N$  of density matrices onto a plane. The case of a large dimension,  $N \gg 1$ , is treated in section 7 jointly with the shadow of random matrices. Finally, in section 8 we provide some concluding remarks and summarize the contribution of this paper.

## 2. Classical and quantum states

Let  $p = \{x_1, x_2, \dots, x_N\}$  be a normalized probability vector, so  $x_i \geq 0$  and  $\sum_{i=1}^N x_i = 1$ . Such a vector represents a *classical state*, and the set  $\mathcal{C}_N$  of all classical states forms an  $(N - 1)$  dimensional regular simplex  $\Delta_{N-1} \subset \mathbb{R}^{N-1}$ . There exist exactly  $N$  *classical pure states*, which correspond to the corners of the simplex. All other classical states can be expressed by a convex combination of pure states and are called *mixed*. Typical mixed states are characterized by the full rank and they form the entire interior of the probability simplex.

In quantum theory one describes a physical system with  $N$  distinguishable states by elements of a complex Hilbert space  $\mathcal{H}_N$  of size  $N$ . Its elements represent pure quantum states,  $|\psi\rangle \in \mathcal{H}_N$ . Quantum states are assumed to be normalized,  $\|\psi\|^2 = \langle\psi|\psi\rangle = 1$ , so they belong to the sphere of dimension  $2N - 1$ . Since one identifies two states, which differ by a global phase only,  $|\psi\rangle \sim |\phi\rangle = e^{-i\alpha}|\psi\rangle$ , the set of all pure quantum states  $\Omega_N$ , which act on  $\mathcal{H}_N$ , is equivalent to the complex projective space,  $\Omega_N = \mathbb{C}P^{N-1}$  [2].

In analogy to the classical case, one also defines mixed quantum states (density matrices) by a convex combination of projectors onto pure states,  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ , where  $p_i > 0$  and  $\sum_i p_i = 1$ . Let us denote the set of all density matrices of order  $N$  by  $\mathcal{Q}_N$ . It contains all density operators which are positive and normalized,

$$\mathcal{Q}_N = \{\rho : \mathcal{H}_N \rightarrow \mathcal{H}_N, \quad \rho^* = \rho, \quad \rho \geq 0, \quad \text{Tr}\rho = 1\}. \quad (3)$$

Since density operators are Hermitian and normalized, this set is  $N^2 - 1$  dimensional. It includes the set of classical states,  $\mathcal{Q}_N \supset \mathcal{C}_N = \Delta_{N-1}$ , as well as the set of pure quantum states,  $\mathcal{Q}_N \supset \Omega_N = \mathbb{C}P^{N-1}$ . We are going to work with the geometry implied by the Hilbert-Schmidt norm of a matrix,  $|A|_{\text{HS}} := \sqrt{\text{Tr}(A^*A)}$ , and the Hilbert-Schmidt distance in the space of matrices,

$$d_{\text{HS}}(A, B) := |A - B|_{\text{HS}} = \sqrt{\text{Tr}(A - B)(A - B)^*}. \quad (4)$$

It will be also convenient to define a real inner-product by setting the polar identity

$$\langle A, B \rangle = \frac{1}{4}|A + B|_{\text{HS}}^2 - \frac{1}{4}|A - B|_{\text{HS}}^2 = \frac{1}{2}[\text{Tr}(A^*B + B^*A)]. \quad (5)$$

If  $A^* = A$  and  $B^* = B$  then  $\langle A, B \rangle = \text{tr}AB$ .

In the set  $\Omega_N$  of quantum pure states one defines the Fubini-Study measure  $\mu_{\text{FS}}$ , which is induced by the Haar measure on  $U(N)$  and is invariant with respect to unitary transformations.

In the case of one-qubit states this measure corresponds to the uniform distribution of points on the Bloch sphere  $S^2$ .

In practice, to generate pure states at random according to the measure  $\mu_{\text{FS}}$  it is sufficient to generate uniformly points at the sphere  $S^{2N-1}$ . One may also select an arbitrary column, (or row) of a random unitary matrix  $U$  distributed according to the Haar measure. It directly gives the set of  $N$  coefficients of the random state in a given basis,  $|\psi\rangle = \sum_{i=1}^N c_i |i\rangle$ . For instance, choosing the first column of  $U$  we set  $c_i = U_{i,1}$  for  $i = 1, \dots, N$ . Alternatively, one may generate  $N$  independent complex random numbers  $z_i$  and renormalise them,  $c_i = z_i / \sqrt{\sum_i |z_i|^2}$ , to obtain the desired distribution [15, 3].

In this work we are going to use the following

**Proposition 1** *Let  $|\psi\rangle \in \Omega_N$  be a random pure state of size  $N$  distributed according to the Fubini-Study measure. If one represents it in an arbitrary fixed basis,  $|\psi\rangle = \sum_{i=1}^N c_i |i\rangle$  then the squared absolute values of the coefficients,  $p_i = |c_i|^2$ , form a probability vector distributed uniformly in the probability simplex  $\Delta_{N-1}$ .*

This is equivalent to the known statement (see e.g. [2]), that the only constraint on the components of a single column of a random unitary matrix  $U$  distributed according to the Haar measure is the normalization condition,  $P(U_{11}, \dots, U_{N1}) \sim \delta\left(1 - \sum_{i=1}^N |U_{i1}|^2\right)$ . This fact directly implies

**Corollary 2** *For any quantum state  $\rho$  define a classical state  $p = \text{diag}(\rho)$ , so  $p_i = \rho_{ii}$ . Then the Fubini-Study measure on the set  $\Omega_N$  of quantum pure states induces by this mapping the uniform measure in the classical probability simplex  $\Delta_{N-1}$ .*

In the case of  $N = 2$  the Fubini-Study measure covers uniformly the Bloch sphere  $S^2$ . Working with the standard polar coordinates,  $(r, \theta, \varphi)$ , we write the element of the volume of the unit sphere as  $dS = d\varphi \sin \theta d\theta = d\varphi d(\cos \theta)$ . The polar angle  $\theta$  is defined with respect to the axis  $z$ , so the projection of a point of the sphere at this axis reads  $z = \cos \theta$ . Hence the Fubini-Study measure implies the uniform distribution  $d(\cos \theta) = dz$  along the one-dimensional set  $\Delta_1$  of  $N = 2$  classical states.

### 3. Numerical range as a projection of the set of quantum states

The set  $\Omega_N = \mathbb{CP}^{N-1}$  of pure states of size  $N$  forms the set of extremal points in  $\mathcal{Q}_N$ . Any mixed state  $\rho \in \mathcal{Q}_N$  can be thus decomposed into a convex mixture of projectors  $|\psi\rangle\langle\psi|$ . The expectation value of an operator  $A$  among a pure state reads  $\langle\psi|A|\psi\rangle = \text{Tr}\rho A$ . Taking into account the convexity of  $W(A)$ , the standard definition (1) of the numerical range of  $A$  can be therefore rewritten as [16]

$$W(A) = \{z : z = \text{Tr}\rho A, \rho \in \mathcal{Q}_N\}. \quad (6)$$

This expression suggests a possible link between numerical range and the structure of the set the  $\mathcal{Q}_N$ . Usually one studies numerical range  $W(A)$  for a given  $A$  [11]. Here we propose to fix the dimension  $N$  and consider the set of all possible numerical ranges of matrices  $A$  of this size to analyze the geometry of quantum states. More precisely, we establish the following facts.

**Proposition 3** *Let  $\mathcal{C}_N$  denote the set of classical states of size  $N$ , which forms the regular simplex  $\Delta_{N-1}$  in  $\mathbb{R}^{N-1}$ . Then the set of similar images of orthogonal projections of  $\mathcal{C}_N$  on a 2-plane is equivalent to the set of all possible numerical ranges  $W(A)$  of all normal matrices  $A$  (such that  $AA^* = A^*A$ ) of order  $N$ .*

**Proposition 4** *Let  $\mathcal{Q}_N$  denote the set of quantum states size  $N$  embedded in  $\mathbb{R}^{N^2-1}$  with respect to Euclidean geometry induced by Hilbert-Schmidt distance. Then the set of similar images of orthogonal projections  $\mathcal{Q}_N$  on a 2-plane is equivalent to the set of all possible numerical ranges  $W(A)$  of (arbitrary) matrices  $A$  of order  $N$ .*

To prove the above propositions we will need an abstract lemma concerning the real inner-product Euclidean spaces.

**Lemma 5** *Suppose  $u_1, u_2, v_0 \in V$ , where  $V$  is a Euclidean vector space (with inner product  $\langle \cdot, \cdot \rangle$  and norm  $|x| = \langle x, x \rangle^{1/2}$ ),  $v_0 \neq 0$  and  $\dim(\text{span}\{u_1, u_2, v_0\}) \geq 2$ . Then there exist real numbers  $\alpha > 0, \gamma_1, \gamma_2$  such that the vectors*

$$v_1 := \frac{1}{\alpha}(u_1 + \gamma_1 v_0), \quad v_2 := \frac{1}{\alpha}(u_2 + \gamma_2 v_0) \quad (7)$$

are normalized and orthogonal,

$$|v_1|^2 = 1 = |v_2|^2, \quad \langle v_1, v_2 \rangle = 0. \quad (8)$$

*Proof.* Let  $u'_i = u_i - \frac{\langle u_i, v_0 \rangle}{|v_0|^2} v_0$ ,  $i = 1, 2$ . By hypothesis  $|u'_1|^2 + |u'_2|^2 > 0$ . For  $i = 1, 2$  set  $c_i := |v_0| \gamma_i + \frac{\langle u_i, v_0 \rangle}{|v_0|}$  so that  $v_i = \frac{1}{\alpha}(u'_i + \frac{c_i}{|v_0|} v_0)$ . The desired equations become

$$|u'_1|^2 + c_1^2 = \alpha^2, \quad |u'_2|^2 + c_2^2 = \alpha^2, \quad \langle u'_1, u'_2 \rangle + c_1 c_2 = 0. \quad (9)$$

Eliminating coefficient  $\alpha$  we arrive at a quadratic equation for  $c_1^2$  or  $c_2^2$ . Set

$$d = (|u'_1|^2 - |u'_2|^2)^2 + 4\langle u'_1, u'_2 \rangle^2,$$

then

$$c_1^2 = \frac{1}{2}(|u'_2|^2 - |u'_1|^2) + \frac{1}{2}\sqrt{d}, \quad (10)$$

$$c_2^2 = \frac{1}{2}(|u'_1|^2 - |u'_2|^2) + \frac{1}{2}\sqrt{d}, \quad (11)$$

$$\text{sign}(c_1 c_2) = -\text{sign}\langle u'_1, u'_2 \rangle, \quad (12)$$

$$\alpha = \left( \frac{1}{2}(|u'_1|^2 + |u'_2|^2) + \frac{1}{2}\sqrt{d} \right)^{1/2}. \quad (13)$$

Recall  $|u'_1|^2 + |u'_2|^2 > 0$  by hypothesis thus  $\alpha > 0$ . There are generally two solutions differing only in the signs of  $c_1$  and  $c_2$ . If  $\langle u'_1, u'_2 \rangle = 0$  then  $\sqrt{d} = ||u'_1|^2 - |u'_2|^2|$ , and one of the three following cases apply:

1.  $|u'_1| > |u'_2| \geq 0$ ,  $c_1 = 0$ ,  $c_2 = \pm \sqrt{|u'_1|^2 - |u'_2|^2}$ ,  $\alpha = |u'_1|$ ;
2.  $|u'_2| > |u'_1| \geq 0$ ,  $c_1 = \pm \sqrt{|u'_2|^2 - |u'_1|^2}$ ,  $c_2 = 0$ ,  $\alpha = |u'_2|$ ;
3.  $|u'_1| = |u'_2| > 0$ ,  $c_1 = 0$ ,  $c_2 = 0$ ,  $\alpha = |u'_1|$ .  $\square$

Note that the formulae (10, 11) for  $c_1$  and  $c_2$  allow us to obtain the constants  $\gamma_1$  and  $\gamma_2$ , which enter eq. (7). The scaling factor  $\alpha = 1$  if and only if  $\langle u'_1, u'_2 \rangle^2 = (1 - |u'_1|^2)(1 - |u'_2|^2)$ ,  $|u'_1|^2 \leq 1$  and  $|u'_2|^2 \leq 1$ .

This lemma implies the following

**Corollary 6** *Suppose  $E \subset \{x \in V : \langle x, v_0 \rangle = 1\}$  and  $u_1, u_2 \in V$  define a linear map  $\Phi : E \rightarrow \mathbb{C}$  by  $x \mapsto \langle x, u_1 \rangle + i \langle x, u_2 \rangle$ . Unless  $u_1, u_2 \in \mathbb{R}v_0$  in which case  $\Phi$  is constant, the map  $\Phi$  is isometrically isomorphic to an orthogonal projection followed by a similarity transformation (dilation and translation).*

*Proof.* By Lemma 5 there exist orthonormal vectors  $v_i = \frac{1}{\alpha}(u_i + \gamma_i v_0)$  for  $i = 1, 2$  and  $\alpha > 0$ . Let  $V_0 = \text{span}\{v_1, v_2\}$ . The orthogonal projection onto  $V_0$  is given by  $\pi x := \langle x, v_1 \rangle v_1 + \langle x, v_2 \rangle v_2$  and this is the general form of a rank 2 orthogonal projection. The linear map  $\theta : a_1 v_1 + a_2 v_2 \mapsto a_1 + i a_2$  is an isometry  $V_0 \rightarrow \mathbb{C}$ . If  $x \in E$  then  $\langle v_0, x \rangle = 1$  and

$$\theta(\alpha \pi x - (\gamma_1 v_1 + \gamma_2 v_2)) = \theta \sum_{i=1}^2 (\langle u_i + \gamma_i v_0, x \rangle - \gamma_i) v_i = \theta \sum_{i=1}^2 \langle u_i, x \rangle v_i = \phi x. \quad (14)$$

□

Now we are ready to prove the main result of this paper, namely Proposition 3 and Proposition 4.

### 3.1. Normal matrices

*Proof of Proposition 3.* Let  $A$  be a normal matrix of order  $N$  with eigenvalues  $\{\lambda_1, \dots, \lambda_N\}$ . With respect to an orthonormal basis of eigenvectors of  $A$  one has  $\sum_{i,j=1}^N \bar{\psi}_i A_{ij} \psi_j = \sum_{i=1}^N \lambda_i |\psi_i|^2$  and the numerical range  $W_A$  is the image of the simplex  $\Delta_{N-1} := \left\{ t \in \mathbb{R}^N : t_i \geq 0 \ \forall i, \sum_{i=1}^N t_i = 1 \right\}$  under the map

$$\Phi : t \mapsto \sum_{i=1}^N t_i \text{Re } \lambda_i + i \sum_{i=1}^N t_i \text{Im } \lambda_i = \langle t, u_1 \rangle + i \langle t, u_2 \rangle, \quad (15)$$

where  $t \in \Delta_{N-1}$ , and  $(u_1)_i = \text{Re } \lambda_i, (u_2)_i = \text{Im } \lambda_i$  for  $1 \leq i \leq N$ . If  $A \neq c\mathbb{1}$  (multiple of the identity, the eigenvalues are all equal) then Lemma 5 and Corollary 6 apply with  $v_0 = (1, \dots, 1)$ , which completes the proof of Proposition 3. □

### 3.2. Non-normal matrices

*Proof of Proposition 4.* The set  $\mathcal{Q}_N$  of quantum states (3) contains Hermitian operators  $\rho$  which can be diagonalized,  $\rho = U D U^*$ . Here  $U$  is unitary while  $D$  is a diagonal matrix with  $d_{ii} \geq 0$  and  $\sum_{i=1}^N d_{ii} = 1$ .

Consider any matrix  $A$  of order  $N$  and write  $\text{Tr} \rho A = \text{Tr} \rho A_1 + i \text{Tr} \rho A_2$  with  $A_1 = \frac{1}{2}(A + A^*)$  and  $A_2 = \frac{1}{2i}(A - A^*)$ . Lemma 5 and Corollary 6 apply now to the map

$$\Phi : \rho \mapsto \text{Tr} \rho A_1 + i \text{Tr} \rho A_2 \quad (16)$$

of the set  $\mathcal{Q}_N$  onto numerical range  $W(A)$  with  $V$  representing the linear space of complex matrices of size  $N$  (or the real subspace of Hermitian matrices), the real inner product (5), and  $v_0 = I, u_1 = A_1, u_2 = A_2$  provided  $A \neq c\mathbb{1}$ .

Thus we have shown that for any matrix  $A$  its numerical range  $W(A)$  is equal to an orthogonal projection of the set of density matrices. To show the converse we may read formulae (7) backwards: the projection of  $\mathcal{Q}_N$  is determined by two orthonormal Hermitian matrices  $V_1$  and  $V_2$ , which then satisfy  $|V_1|_{\text{HS}} = |V_2|_{\text{HS}} = 1$  and  $\text{Tr}(V_1 V_2) = 0$ . Set  $A = V_1 + i V_2$ , which gives now the required matrix such that  $W(A)$  is equal to the desired projection. In this way a link between numerical ranges of generic matrices of order  $N$  and projections of the set  $\mathcal{Q}_N$  onto a two-plane is established and Proposition 4 is proved. □

To obtain explicit formulae for the similarity transformation corresponding to an arbitrary matrix  $A$  of order  $N$  define three traceless matrices

$$B = A - \frac{\text{Tr} A}{N} I, \quad B_1 = \frac{1}{2}(B + B^*), \quad B_2 = \frac{1}{2i}(B - B^*). \quad (17)$$

The latter two represent vectors in the Hilbert–Schmidt space and correspond to  $u'_1, u'_2$  in Lemma 5. Making use of the Hilbert–Schmidt norm we compute the required coefficients for a given traceless matrix  $B$

$$d = \text{Tr} B^2 \text{Tr} B^{*2} = |\text{Tr} B^2|^2, \quad \alpha = \left( \frac{1}{2} \text{Tr}(BB^*) + \frac{1}{2} |\text{Tr} B^2| \right)^{1/2}, \quad (18)$$

$$c_1^2 = -\frac{1}{4} (\text{Tr} B^2 + \text{Tr} B^{*2}) + \frac{1}{2} |\text{Tr} B^2|, \quad c_2^2 = \frac{1}{4} (\text{Tr} B^2 + \text{Tr} B^{*2}) + \frac{1}{2} |\text{Tr} B^2|, \quad (19)$$

and  $\text{sign}(c_1 c_2) = -\text{sign}\langle u'_1, u'_2 \rangle = -\text{sign}(\text{Im Tr} B^2)$ .

#### 4. Numerical shadow and quantum states

The projectors  $|\psi\rangle\langle\psi|$  onto pure states form extremal points of the set  $\mathcal{Q}_N$  of quantum states, hence the shape of a projection of the set  $\Omega_N$  of pure states onto a given plane coincides with the shape of the projection of the set of density matrices on the same plane. As shown in the previous section this set is equal to the numerical range  $W(A)$  of a matrix  $A$  of size  $N$ , which determines the projection.

However, the differences appear if one studies not only the support of the projection but also the corresponding probability measure. A measure  $P_A(z)$  determined by the numerical shadow (2) is induced by the Fubini–Study measure on the set  $\Omega_N$  of pure state. Thus the standard numerical shadows of various matrices of size  $N$  can be interpreted as a projection of the complex projective space,  $\Omega_N = \mathbb{C}P^{N-1}$ , onto a plane. Before discussing in detail the cases of low dimensions, let us present here some basic properties of the numerical shadow [13] (also called the *numerical measure* [14]).

1. By construction the distribution  $P_A(z)$  is supported on the numerical range of  $W(A)$  and it is normalized,  $\int_{W(A)} P_A(z) d^2 z = 1$ .
2. The (numerical) shadow is unitarily invariant,  $P_A(z) = P_{UAU^*}(z)$ . This is a consequence of the fact that the integration measure  $d\mu(\psi)$  is unitarily invariant.
3. For any normal operator  $A$  acting on  $\mathcal{H}_N$ , such that  $AA^* = A^*A$ , its shadow covers the numerical range  $W(A)$  with the probability corresponding to a projection of a *regular*  $N$ -simplex of classical states  $\mathcal{C}_N$  (embedded in  $\mathbb{R}^{N-1}$ ) onto a plane.
4. For a non-normal operator  $A$  acting on  $\mathcal{H}_N$ , its shadow covers the numerical range  $W(A)$  with the probability corresponding to an orthogonal projection of the complex projective manifold  $\Omega_N = \mathbb{C}P^{N-1}$  onto a plane.
5. For any two operators  $A$  and  $B$  acting on  $\mathcal{H}_N$ , the shadow of their tensor product does not depend on the order,

$$P_{A \otimes B}(z) = P_{B \otimes A}(z). \quad (20)$$

To show this property define a unitary swap operator  $S$  which acts on a composite Hilbert space and interchanges the order in the tensor product,  $S(|z\rangle \otimes |y\rangle) = |y\rangle \otimes |z\rangle$ . Thus  $\langle x|A \otimes B|x\rangle = \langle x|S^*B \otimes AS|x\rangle$ , and since  $S$  is unitary it does not influence the numerical shadow induced by the unitarily invariant Fubini–Study measure on complex projective space.

#### 4.1. One-qubit states, $N = 2$

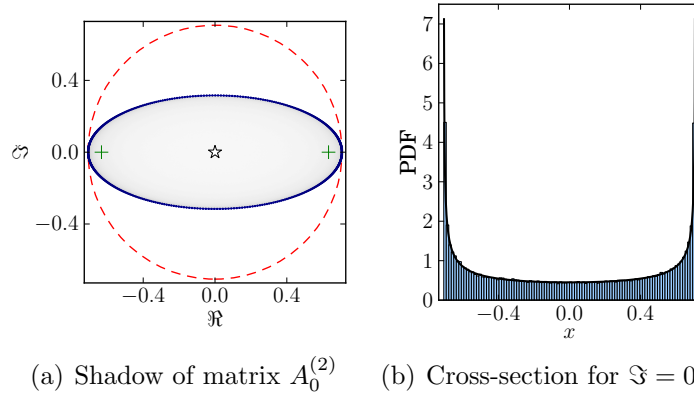
Analysis of the numerical shadow is particularly simple in the case of matrices of order  $N = 2$ . The spectrum of the operator  $A$  consists of two complex numbers,  $\sigma(A) = \{\lambda_1, \lambda_2\}$ .

In the case of a normal matrix  $A$  the numerical range  $W(A)$  forms the closed interval  $[\lambda_1, \lambda_2]$ , and the numerical shadow  $P_A(z)$  covers this interval uniformly [13].

If the matrix  $A$  is non-normal the numerical range forms an elliptical disk with  $\lambda_1, \lambda_2$  as focal points and minor axis,  $d = \sqrt{\text{Tr}AA^* - |\lambda_1|^2 - |\lambda_2|^2}$ . For a simple proof of this 1932 result of Murnaghan [17] see the note by Li [18]. In this generic case the numerical shadow is given by the probability distribution obtained by the projection of the hollow Bloch sphere of one-qubit pure states onto a plane [13]. In particular, the cross-section of the numerical shadow supported in an interval  $x \in [0, 1]$  is given by the arcsine distribution,  $P(x) = 1/(\pi\sqrt{x(1-x)})$ . Non-normal case is shown in Fig.1, obtained for a matrix,

$$A_0^{(2)} = a_0 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

For simplicity we have selected the centred matrix such that  $\text{Tr}A = 0$  so that one has  $B = A$  in eq. (17). The normalization constant  $a_0 = \sqrt{2/5}$  is chosen in such a way that the scaling constant defining the projection in (18) is set to unity,  $\alpha = 1$ , so the shadow of the set of quantum states is shown in its 'natural size': The distance between both eigenvalues,  $l = 2a_1 = \sqrt{2}$ , is equal to the diameter of the Bloch ball,  $2R_2 = \sqrt{2}$ .



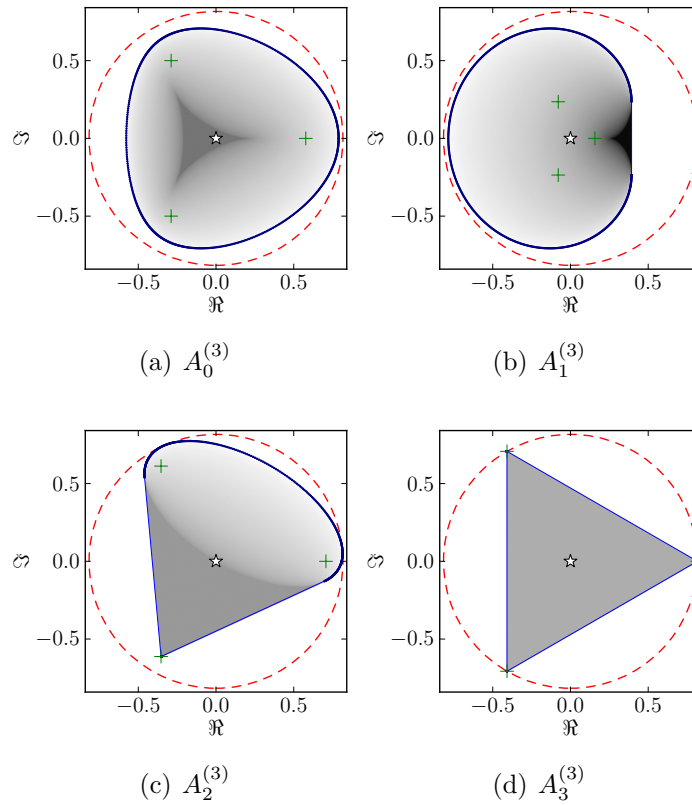
**Figure 1.** Projection of the set  $\Omega_2$  of one-qubit states generated by the numerical shadows of operators of order  $N = 2$  a) numerical shadow of generic matrix  $A_0^{(2)}$  with an elliptical support. Eigenvalues are denoted with crosses and dashed circle of radius  $R_2 = \sqrt{2}/2$  denotes the diameter of the Bloch ball. Numerically obtained histogram is plotted in black, analytical plot is blue. Plot is done for matrix translated in such a way that its trace ( $\star$ ) is equal to zero and suitably rescaled. b) Histogram of cross-section of the shadow supported in the interval  $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ , solid line represents a probability density function of the arcsine distribution  $P(x) = \left(\pi\sqrt{\frac{1}{2} - x^2}\right)^{-1}$ .

#### 4.2. One-qutrit states, $N = 3$

The structure of the numerical range for  $N = 3$  was analyzed in detail by Keeler et al. [19]. Numerical range of a matrix  $A$  of order  $N = 3$  with spectrum  $\lambda_1, \lambda_2, \lambda_3$  forms:



- a) a compact set of an 'ovular' shape with three eigenvalues in its interior;
- b) a compact set with *one* flat part – e.g. the convex hull of a cardioid;
- c) a compact set with *two* flat parts – e.g. the convex hull of an ellipse and a point outside it;
- d) triangle. For any **normal** matrix  $A$  its numerical range is equal to the triangle spanned by the spectrum,  $W(A) = \Delta(\lambda_1, \lambda_2, \lambda_3)$ . In the latter case the numerical shadow can be verbally interpreted as the shadow of the set  $\mathcal{C}_3$  of  $N = 3$  classical states – a uniformly covered equilateral triangle  $\Delta_2$ .



**Figure 2.** Projections of the set  $\Omega_3$  of one-qutrit states generated by the numerical shadows of operators of order  $N = 3$ ; a) a generic matrix  $A_0^{(3)}$  with an oval-like numerical shadow, b)  $A_1^{(3)}$  with one flat part of the boundary  $\partial W$  of the numerical range, c)  $A_2^{(3)}$  a simple sum with two flat parts of  $\partial W$ , d) a diagonal normal matrix  $A_3^{(3)}$  with numerical range equal to the triangle of eigenvalues, represented by (+). Dashed circle of radius  $R_3$  represents projection of the sphere in which  $\Omega_3$  is inscribed. All plots are done for matrices translated in such a way that their trace ( $\star$ ) is equal to zero and suitably rescaled.

The four classes of  $N = 3$  numerical ranges are illustrated in figure 2. It shows the numerical shadow supported on the corresponding numerical range, obtained for

$$A_0^{(3)} = a_0 \begin{bmatrix} 1 & 1 & 1 \\ 0 & \omega_3 & 1 \\ 0 & 0 & \omega_3^2 \end{bmatrix}, \quad A_1^{(3)} = a_1 \begin{bmatrix} 5 - 3i & 0 & 6 \\ 0 & 5 + 3i & 6 \\ -6 & -6 & -10 \end{bmatrix},$$

$$A_2^{(3)} = a_2 \begin{bmatrix} 1 & 1 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix}, \quad A_3^{(3)} = a_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix}.$$

The symbol  $\omega_k$  denotes the  $k$ -th root of unity, so  $\omega_3 = \exp(i2\pi/3)$ . As before the matrices are chosen to be traceless, so  $B = A$  in (17) and the shadows are centered. Furthermore, the normalization constants are designed to assure that the scaling constant in eq. (18) in every case is set to unity,  $\alpha = 1$ , so the Figure shows images of the set of quantum states in its natural size. For instance in the case of the diagonal matrix  $A_3^{(3)}$  the prefactor reads  $a_3 = \sqrt{2/3}$ , so that the eigenvalues are located at the distance  $\sqrt{2/3}$  from the origin. This is just the radius  $R_3$  of the sphere in which the set  $\mathcal{Q}_3$  is inscribed.

The study of the geometry of the numerical range was initiated by Kippenhahn [20] and later developed by Fiedler [21]. In more recent papers [22, 23] the differential topology and projection aspects of numerical range were investigated. In particular it was shown [22] that the numerical range of a generic matrix  $A$  of order three pertains to the class a) above, as the boundary of  $W(A)$  does not contain intervals. Critical lines inside the range, analysed in [22, 23] where shown to influence the structure of the numerical shadow [13]. Thus we may now relate the critical lines with the geometry of complex projective spaces projected onto a plane.

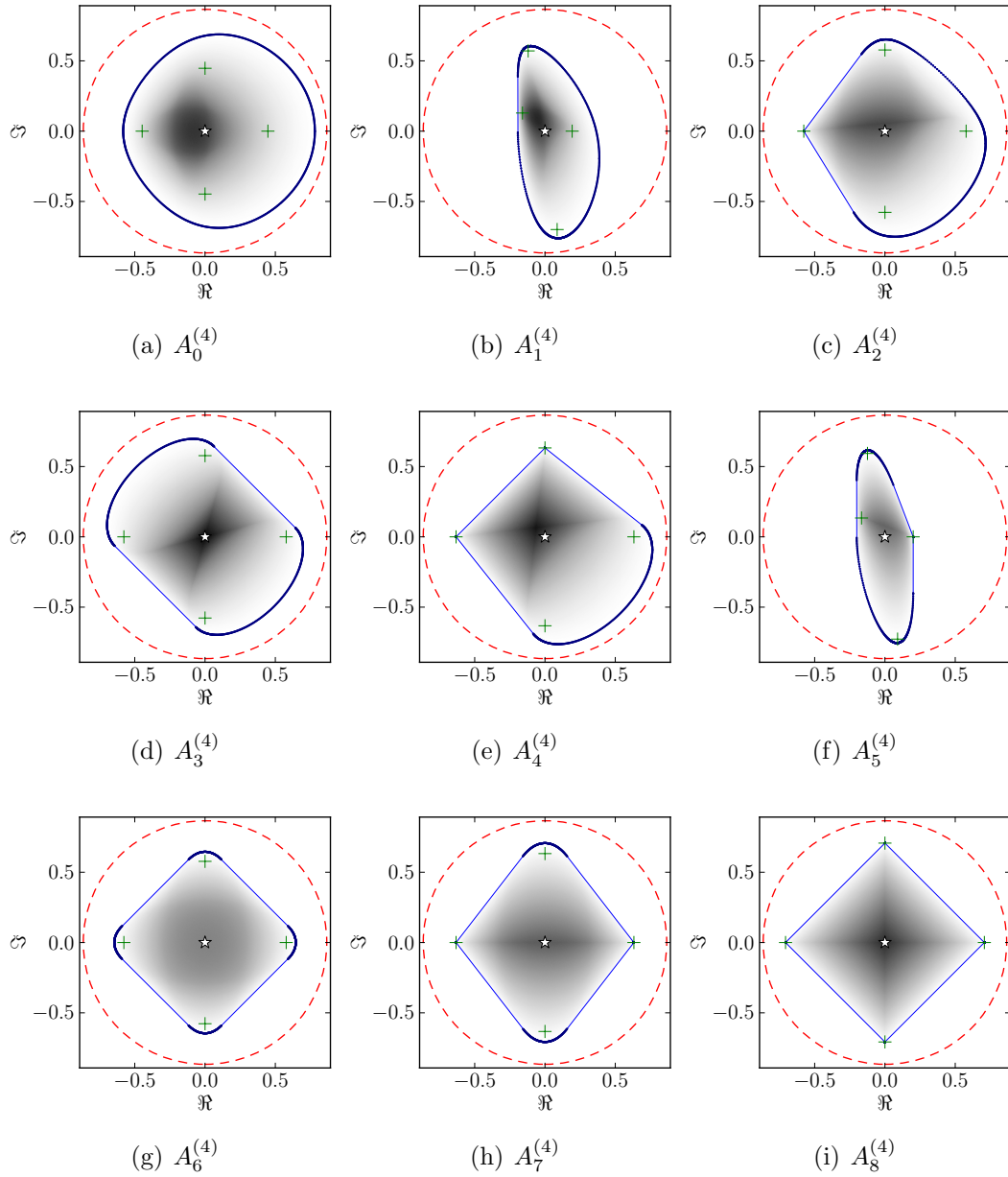
In the one-qutrit case  $N = 3$  obtained probability distributions can be interpreted as images of the set of pure states  $\Omega_3 = \mathbb{C}P^2$  on the plane. Although it is not so simple to imagine the structure of the complex projective space [24], some experience is gained by studying numerical shadows of various non-normal matrices of size 3.

#### 4.3. Four-level systems, $N = 4$

Various shapes of the numerical range for matrices of size  $N = 4$  correspond to various projections of the set  $\mathcal{Q}_4$  of quantum states of size four. As in the case of the qutrit we analyse numerical shadows of traceless matrices normalized such that the scaling constant  $\alpha$  is set to unity.

Even though several results on geometry of numerical range for  $N = 4$  are available [25, 26], complete classification of numerical ranges in this case is still missing. To provide an overview of the possible structure of the numerical shadow we analysed the following matrices of order four,

$$\begin{aligned} A_0^{(4)} &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & i & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -i \end{bmatrix}, \quad A_1^{(4)} = \begin{bmatrix} i & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 1-i & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad A_2^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & i & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, \\ A_3^{(4)} &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & i & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, \quad A_4^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, \quad A_5^{(4)} = \begin{bmatrix} i & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 1-i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ A_6^{(4)} &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & i & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, \quad A_7^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, \quad A_8^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}. \end{aligned}$$



**Figure 3.** Projections of the set  $\Omega_4$  of  $N = 4$  quantum states emerging as numerical shadows of appropriately normalized operators of size 4: a) a generic matrix  $A_0^{(4)}$  with an oval-like numerical range  $W(A)$ , b)  $A_1^{(4)}$  with one flat part of the boundary  $\partial W$  of the numerical range, c)  $A_2^{(4)}$  being a simple sum  $3 \oplus 1$  with two flat parts of  $\partial W$ , d)  $A_3^{(4)}$  – a simple sum  $2 \oplus 2$  with two flat parts of  $\partial W$ , e)  $A_4^{(4)}$  three flat parts of  $\partial W$  connected with corners and one oval-like part, f)  $A_5^{(4)}$  three flat parts of  $\partial W$  with only one corner and two oval-like parts, g)  $A_6^{(4)}$  a simple sum  $2 \oplus 2$ , with four flat parts of  $\partial W$ , h)  $A_7^{(4)}$  pair of flat parts of  $\partial W$  connected with a corner connected with two oval-like parts, i) a diagonal normal matrix  $A_8^{(4)}$  with numerical range  $W$  equal to the convex hull of eigenvalues denoted by (+). All plots are done for matrices translated in such a way that their trace ( $\star$ ) is equal to zero and suitably rescaled.

Numerical shadows of these representative of each class of  $N = 4$  matrices are shown in figure 3. The pictures can be interpreted as projections of the 6-dimensional complex projective space  $\mathbb{CP}^3$  onto a plane. Making use of formula (18) we find that the normalization constant for the last example  $A_8^{(4)}$  reads  $a_8 = 1/\sqrt{2}$ . Thus the diameter of the shadow,  $2a_4 = \sqrt{2}$ , coincides in this case with the Hilbert-Schmidt distance between any two orthogonal pure states in  $\mathcal{Q}_4$ . The dashed circle of radius  $R_4 = \sqrt{3}/2$  represents projection of the sphere into which the set  $\mathcal{Q}_4$  can be inscribed.

## 5. Unitary dynamics projected inside the numerical shadow

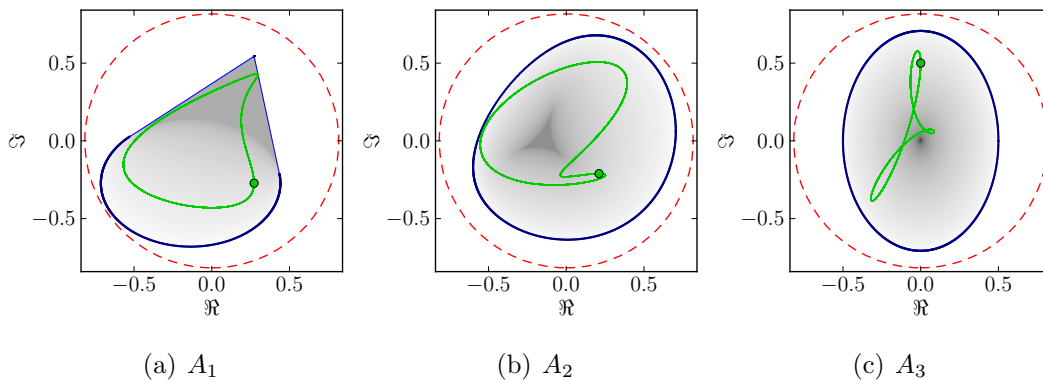
As the numerical range and the numerical shadow give us an opportunity to observe the structure of the space of quantum states, it is possible to apply these tools to investigate quantum dynamics. A unitary time evolution of a quantum system is governed by the Hamiltonian operator  $H$  (i.e. a self-adjoint operator representing the total energy of the system), which leads to  $U(t) = \exp(-iHt)$ .

Note that eigenvalues of the Hamiltonian determine the cyclicity of the trajectory. The trajectory is periodic iff eigenvalues of the Hamiltonian are commensurable. The period in this case is given by least common multiple of the eigenvalues.

Let us consider a three-level system (qutrit). For concreteness, we choose the Hamiltonian

$$H = \begin{bmatrix} -1 & -1-i & 1 \\ -1+i & 0 & 1+i \\ 1 & 1-i & 1 \end{bmatrix} \quad (21)$$

and select an initial pure state of the system as  $|\psi(0)\rangle = |0\rangle \in \mathcal{H}_3$ . The state of the system at some specific time  $t$  is described by the transformed state  $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ .



**Figure 4.** Shadow of the operators  $A_1$ ,  $A_2$  and  $A_3$  of order three serves as a background for the trajectory representing the unitary dynamics defined by  $U = \exp(-iHt)$  with  $H$  given by (21) with the initial state  $|\phi(0)\rangle$  (marked by circle in the picture). All plots are done for matrices translated in such a way that their trace is equal to zero and suitably rescaled.

In order to use the numerical shadow to study the time evolution of the system, one needs to choose an arbitrary  $3 \times 3$  non-hermitian matrix. To get some information on the

dynamics in the space of pure states of a qutrit and to observe it from different points of view, we selected the following matrices

$$A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & i & 1 \\ 0 & 0 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} i & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix}. \quad (22)$$

For each of these auxiliary matrices, the quantum dynamics can be now visualized as a trajectory in the complex plane defined by a parametric equation,

$$z(t) := \langle \psi(t) | A | \psi(t) \rangle = \langle \psi(0) | A' | \psi(0) \rangle, \quad (23)$$

where the unitarily transformed matrix reads  $A' = U^* A U = e^{iHt} A e^{-iHt}$ . The time evolution of the initial state  $|\psi(0)\rangle = (1, 0, 0)^T \in \mathcal{H}_3$ , generated by the Hamiltonian (21) is shown in Fig. 4 from three different perspectives determined by matrices (22).

### 5.1. Identical trajectories

For given matrix  $A$  and hermitian matrix  $H$  a unitary time evolution induces a path in the numerical range  $\Lambda_A$  given by

$$\xi^* e^{-iHt} A e^{iHt} \xi, \text{ for given starting point } \xi. \quad (24)$$

In the mixed state scenario the trajectory is given by

$$\text{tr } \rho e^{-iHt} A e^{iHt}, \text{ for given starting point } \rho. \quad (25)$$

The question one may pose is: under what conditions for two different starting points  $\rho_0$  and  $\rho_1$  trajectories on numerical range of  $A$  are identical:

$$\text{tr } \rho_0 e^{-iHt} A e^{iHt} = \text{tr } \rho_1 e^{-iHt} A e^{iHt}, \quad t \in \mathbb{R}. \quad (26)$$

To convince yourself that such a situation may occur consider the daily rotation of the earth around its axis. Choosing two initial points at the same meridian on opposite sides of the equator (say close to Cairo and Durban in Africa), we see that the trajectories they generate after projecting onto the equatorial plane do coincide. This is because the dynamics, both initial points and the kind of the projection, are chosen in a special way and satisfy certain constraints. To characterize these constraints in a general setting we start with the following definitions.

For given matrix  $A$ , let  $X_A = \{B \in M_N^H(\mathbb{C}) : \text{tr } B = 0, \text{tr } BA = 0\}$ . We also define the set  $H_A$

$$H_A = \{H \in M_N^H(\mathbb{C}) : \forall t > 0, B \in X_A \text{ we have } \text{Ad}_{e^{iHt}}(B) \in X_A\}, \quad (27)$$

where  $\text{Ad}$  is the adjoint mapping given by  $\text{Ad}_C(B) = C B C^{-1}$ .

Now we can state fact concerning identical trajectories.

**Lemma 7** *Trajectories  $\text{tr } \rho_0 e^{-iHt} A e^{iHt}$  and  $\text{tr } \rho_1 e^{-iHt} A e^{iHt}$  for  $t \in \mathbb{R}$  are identical if and only if  $\rho_0 - \rho_1 \in X_A$  and  $H \in H_A$ .*

The definition of  $H_A$  is somehow complicated, here we put reasoning which presents it in a simpler form. We have following property  $\text{Ad}_C = e^{\text{ad}_C}$ , where  $\text{ad}_C(B) = [C, B]$  (see e.g. [34]). Using above we can state the following lemma.

**Lemma 8** *Hermitian matrix  $H$  is an element of  $H_A$ , if and only if, for all  $B \in X_A$  we have  $\text{ad}_{iH}(B) \in X_A$ .*

*Proof.* If  $\text{ad}_{iH}(B) \in X_A$  for all  $B \in X_A$ , then by iterating we have that  $\text{ad}_{iH}^k(B) \in X_A$  for  $k = 0, 1, \dots$ . Since  $X_A$  is a linear space we obtain, that

$$\text{Ad}_{e^{itH}}(B) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \text{ad}_{iH}^k(B) \in X_A. \quad (28)$$

On the other hand if  $\text{Ad}_{e^{itH}}(B) \in X_A$ , then using the fact that

$$i[H, B] = \lim_{t \rightarrow 0} \frac{e^{iHt} B e^{-iHt} - B}{t}, \quad (29)$$

and the continuity of the function  $X \mapsto \text{tr} XA$  we obtain the result.  $\square$

Note, that the condition  $i[H, B] \in X_A$  can be stated as  $\text{tr} H[A, B] = 0$ , this follows from the cyclicity of trace. The linear space  $X_A$  is a real  $N^2 - 1 - d(A)$  dimensional space, where  $d(A) = \dim(\{\Re(A), \Im(A)\})$ , where  $\Re(A) = \frac{1}{2}(A + A^*)$  and  $\Im(A) = \frac{1}{2i}(A - A^*)$ . Thus in the generic case  $X_A$  has dimension  $N^2 - 3$ . The set  $H_A$  forms a real subspace of hermitian matrices orthogonal to sum of two real subspaces ( $\text{ad}_{i\Re(A)}(X_A)$  and  $\text{ad}_{i\Im(A)}(X_A)$ ),

$$H_A = (\text{ad}_{i\Re(A)}(X_A) \cup \text{ad}_{i\Im(A)}(X_A))^{\perp}, \quad (30)$$

where  $^{\perp}$  denotes the orthogonal component in the real space of hermitian matrices.

## 6. Mixed states numerical shadow

The standard numerical shadow (2) of matrix  $A$  is defined by choosing randomly a pure state  $\rho = |\psi\rangle\langle\psi|$  with respect to the unitarily invariant, natural measure on the set of pure states, and taking the expectation value  $\text{Tr} A\rho$ . However, one may also consider an expression analogous to (6) and use it with a different measure in the set  $\mathcal{Q}_N$  of mixed states. More precisely, we introduce the *mixed states numerical shadow* of  $A$  with respect to a measure  $\mu$ ,

$$P_A^{\mu}(z) := \int_{\mathcal{Q}_N} d\mu(\rho) \delta(z - \text{Tr} \rho A). \quad (31)$$

The measure  $\mu$  defined on the set  $\mathcal{Q}_N$  of mixed states of size  $N$  is supposed to be unitarily invariant. For instance, we will use the family of *induced measures*  $\mu_K$  obtained by taking a random pure state  $|\xi\rangle \in \mathcal{H}_N \otimes \mathcal{H}_K$  and generating a mixed state by partial trace over the  $K$ -dimensional subsystem,  $\rho = \text{Tr}_K |\xi\rangle\langle\xi|$ . Since the pure states  $|\xi\rangle$  are generated randomly the unitary matrices determining the eigenvectors of  $\rho$  are distributed according to the Haar measure on  $U(N)$ . The probability distribution of the eigenvalues  $\lambda_i$  of the random mixed state  $\rho$  of size  $N$  obtained in this way reads

$$P_{N,K}(\lambda) = C_{N,K} \delta \left( 1 - \sum_{i=1}^N \lambda_i \right) \prod_{i=1}^N \lambda_i^{K-N} \prod_{i < j} (\lambda_i - \lambda_j)^2. \quad (32)$$

It is assumed here that  $K \geq N$  and the normalization constants  $C_{N,K}$  are given in [27]. In the symmetric case,  $K = N$ , the above formula simplifies and the measure  $\mu_N$  coincides with the flat Hilbert–Schmidt measure, induced by the metric (4). In the opposite case  $K < N$ , the joint probability density function is given by (32) with exchanged parameters  $N \leftrightarrow K$ .

Consider now a pure state  $|\xi\rangle$  on the bi-partite  $N \times K$  system. It can be represented in its Schmidt decomposition [2],

$$|\xi\rangle = \sum_{i=1}^{\min\{N,K\}} \sqrt{\lambda_i} |e_i\rangle \otimes |f_i\rangle, \quad (33)$$

where  $\{|e_i\rangle\}_{i=1}^N$  is an orthonormal basis in  $\mathcal{H}_N$  while  $\{|f_i\rangle\}_{i=1}^K$  is an orthonormal basis of  $\mathcal{H}_K$ . Taking a partial trace of the projector  $|\xi\rangle\langle\xi|$  over the  $K$  dimensional system we see that the spectrum of the resulting mixed state  $\rho$  coincides with the set of the Schmidt coefficients  $\{\lambda_i\}$  of the pure state  $|\xi\rangle$ . Thus formula (32) describes the distribution of the Schmidt coefficients of a pure state  $|\xi\rangle$  drawn randomly according to the uniform distribution on the sphere  $S^{NK-1}$ . By construction of the Schmidt decomposition of a random state  $|\xi\rangle$ , the vectors  $|e_i\rangle$  and  $|f_i\rangle$  can be considered as columns of unitary matrix in  $U(N)$  and  $U(K)$  respectively, distributed according to the Haar measure on the unitary group. A simple calculation shows that

$$\begin{aligned} \langle\xi|(A \otimes \mathbb{1}_K)|\xi\rangle &= \sum_{i,j=1}^{\min\{N,K\}} \sqrt{\lambda_i \lambda_j} \langle e_i|A|e_j\rangle \langle f_i|f_j\rangle \\ &= \sum_{i=1}^{\min\{N,K\}} \lambda_i \langle e_i|A|e_i\rangle = \sum_{i=1}^{\min\{N,K\}} \lambda_i (U^\dagger A U)_{i,i}, \end{aligned} \quad (34)$$

where  $U$  is a unitary matrix distributed according to the Haar measure on  $U(N)$ .

These considerations imply that the shadow of  $A \otimes \mathbb{1}_K$  is a mixture of diagonal elements of  $A$  in a random basis, given by the sum  $\sum_{i=1}^{\min\{N,K\}} \lambda_i (U^\dagger A U)_{i,i}$ . As before  $U$  stands for a random unitary matrix of size  $N$  while the joint probability distribution function of the coefficients  $\lambda_i$  is given by (32). This proves that the mixed states shadow of  $A$  with respect to the induced measure  $\mu_K$  coincides with the standard numerical shadow of the extended operator  $A \otimes \mathbb{1}_K$ ,

$$P_A^{\mu_K}(z) = P_{A \otimes \mathbb{1}_K}(z) = P_{\mathbb{1}_K \otimes A}(z). \quad (35)$$

The last equality follows from the property (20). In the most important case,  $K = N$ , the induced measure  $\mu_N$  is equivalent to the Euclidean (flat) measure in  $\mathbb{R}^{N^2-1}$ , corresponding to the Hilbert-Schmidt distance (4). Thus the projection of the 'full' set  $\mathcal{Q}_N$  of mixed quantum states on the plane determined by a given matrix  $A$  of size  $N$  is equivalent to the standard shadow of an extended operator  $A \otimes \mathbb{1}_N$ . In the case of  $N = 2$  this is visualized in figure 5c), in which the shadow of the full Bloch ball  $\mathcal{Q}_2$  can be compared with the shadow of the hollow Bloch sphere  $\Omega_2 = S^2$ , displayed in figure 5b).

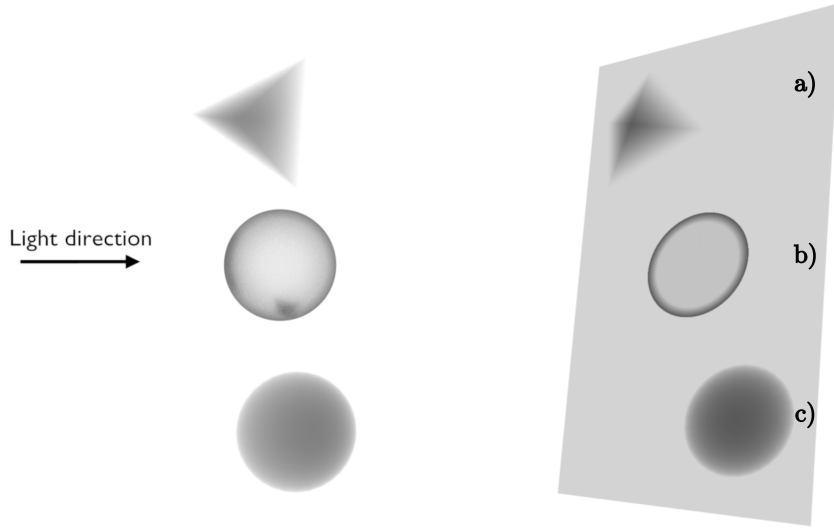
Note that for  $K = 1$  the induced measure  $\mu_1$  is supported on the set  $\Omega_N$  of pure states only and coincides with the Fubini-Study measure, so formula (31) with  $\mu = \mu_1$  reduces to the standard definition (2) of the pure states numerical shadow.

## 7. Large $N$ limit and random matrices

It is instructive to analyse the numerical shadow of a random matrix in the limit of large matrix dimension  $N$ . Let us consider two cases of the problem: the shadow of a random density matrix  $\sigma$  generated according to the induced measures [27] and the shadow of random unitary matrix  $U$  distributed with respect to the Haar measure on  $U(N)$ .

The numerical shadow of random matrix which is distributed with unitarily invariant measure is related to the distribution of its arbitrary diagonal element in a fixed basis. In this section we consider the measures induced by partial trace and the Haar measure on the unitary group, which are unitarily invariant. Let us start with the following

**Lemma 9** *Let  $A$  be a random square matrix of order  $N$  distributed according to a unitarily invariant measure. Let  $|x\rangle$  be a random pure state of size  $N$  generated according to the Fubini-*



**Figure 5.** Sketch of a projections onto a two-plane of a) the set  $\mathcal{C}_4 = \Delta_3$  of  $N = 4$  classical states onto the quadrangle formed by the numerical range  $W$  of a normal matrix  $A$  of order 4; b) the set  $\Omega_2 = S^2$  of one-qubit pure quantum states onto a disk formed by the numerical range of a non-normal Jordan matrix  $J_2$  of order 2; c) a mixed-states numerical shadow of  $J_2$ , corresponding to the projection of the full 3-dimensional Bloch ball onto a plane, is equal to the standard, pure states shadow of an extended matrix  $J_2 \otimes \mathbb{1}_2$ . The picture is plotted using perspective.

Study measure on  $\Omega_N = \mathbb{C}P^{N-1}$ . Then the expectation value has the same distribution as the matrix element  $A_{1,1}$

$$P(\langle x|A|x\rangle) = P(A_{1,1}). \quad (36)$$

*Proof.* Since  $|x\rangle$  is a random pure state, thus  $|x\rangle \sim U|0\rangle$ , where  $|0\rangle$  is an arbitrary fixed state while  $U$  is a random unitary matrix of size  $N$ . Now we write

$$P(\langle x|A|x\rangle) = P(\langle 0|U^\dagger A U|0\rangle) = P(A_{1,1}). \quad (37)$$

The last equality follows from invariance of the distribution  $P(A)$  with respect to unitary transformations.  $\square$

### 7.1. Shadow of random quantum state

Consider a random density matrix  $\sigma$  of size  $N$  generated with respect to the Hilbert-Schmidt measure  $\mu_{HS}$ , so that its eigenvalues  $\lambda_i$  are distributed according to eq. (32) with  $K = N$  [27]. The diagonal elements of  $\sigma$  are of the form

$$\sigma_{ii} = \frac{\sum_{j=1}^N (\xi_{ij}^2 + \eta_{ij}^2)}{\sum_{j,k=1}^N (\xi_{jk}^2 + \eta_{jk}^2)}, \quad (38)$$

where  $\xi_{ij}$  and  $\eta_{ij}$  are independent, identically distributed random variables with normal distribution  $\mathcal{N}(0, 1)$ . Basic properties of the Gamma distribution  $\Gamma(a, b)$  [28] imply that

$$\sigma_{ii} = \frac{G_1}{G_1 + G_2}, \quad (39)$$



where  $G_1$  and  $G_2$  are stochastically independent variables distributed according to the Gamma distribution  $\Gamma(N, 2)$  and  $\Gamma(N(N-1), 2)$  respectively. Therefore the diagonal elements of a random matrix  $\sigma$  generated according to the measure  $\mu_{HS}$  are described by the Beta distribution with parameters  $\{N, N(N-1)\}$ .

The same reasoning can also be used for a general class of induced measures (32) parameterized by the size  $K$  of the auxiliary subsystem. In this case the diagonal elements of a density matrix  $\sigma \in \mathcal{Q}_N$  generated with respect to the measure  $\mu_{N,K}$  are distributed according to the Beta distribution with parameters  $\{K, K(N-1)\}$ .

Using the above reasoning and Lemma 9 we get the following

**Lemma 10** *The numerical shadow of a random matrix  $\sigma$  generated with respect to the induced measure  $\mu_{N,K}$  is given by the Beta distribution with parameters  $\{K, K(N-1)\}$ , which can be expressed in terms of the Beta function,*

$$P_\sigma(r) = \frac{1}{B(K, K(N-1))} (1-r)^{K-1} r^{K(N-1)-1}, \quad 0 \leq r \leq 1. \quad (40)$$

## 7.2. Random unitary matrices

Let us now consider a random unitary matrix  $U$  distributed according to the Haar measure.

**Lemma 11** *The numerical shadow of a Haar random unitary matrix is supported in the unit disk. This distribution is invariant with respect to rotations, and  $|\langle x|U|x \rangle|^2$  is distributed according to the Beta distribution with parameters  $\{1, N-1\}$ .*

*Proof.* Random unitary matrix distributed with Haar measure can be generated using the QR decomposition of matrices pertaining to the Ginibre ensemble [29]. The QR factorization can be realized by a Gram-Schmidt orthogonalization procedure. Then the element  $U_{1,1}$  of the generated unitary matrix reads

$$U_{1,1} = \frac{A_{1,1}}{\sqrt{\sum_{i=1}^N |A_{i,1}|^2}}, \quad (41)$$

where  $A$  is a nonhermitian random matrix from the Ginibre ensemble. Therefore

$$|U_{1,1}|^2 = \frac{\xi_{1,1}^2 + \eta_{1,1}^2}{\sum_{i=1}^N (\xi_{i,1}^2 + \eta_{i,1}^2)}, \quad (42)$$

where  $\xi_{ij}$  and  $\eta_{ij}$  are independent, identically distributed random variables with normal distribution  $\mathcal{N}(0, 1)$ . Thus  $|U_{1,1}|^2$  has the Beta distribution with parameters  $\{1, N-1\}$  and using Lemma 9 we arrive at the desired result.  $\square$

## 8. Concluding remarks

Our study may be briefly summarized by the following observation. The *numerical shadow* of a normal operator acting on  $\mathcal{H}_N$  reflects the structure of the set of (mixed) classical states, which belong to the probability simplex  $\Delta_{N-1}$ , while investigation of numerical shadows of *non-normal operators* provides information about the set  $\mathcal{Q}_N$  of quantum states of size  $N$ .

In particular we have shown that the set of orthogonal projections of the set  $\mathcal{Q}_N$  of density matrices onto a two-plane is equivalent, up to shift and rescaling, to the set of all possible numerical ranges  $W(A)$  of matrices of order  $N$ . The numerical shadow of  $A$  forms a probability distribution on the plane, supported in  $W(A)$ , which corresponds to the ‘shadow’ of the complex projective space  $\mathbb{CP}^{N-1}$  covered uniformly according to the Fubini-Study

measure, and projected onto the plane. Another probability distribution in  $W(A)$  is obtained if one projects onto this plane entire convex set  $\mathcal{Q}_N$  of density matrices. If this set is covered uniformly with respect to the Hilbert-Schmidt (Euclidean) measure, an explicit expression for this distribution is derived. In this way the analysis of numerical ranges and numerical shadows of matrices of a fixed size  $N$  contributes to our understanding of the intricate geometry of the set  $\mathcal{Q}_N$  of quantum states [2].

The numerical range [30] and its generalizations [31, 16] found several application in various problems of quantum information theory. In analogy to the *product numerical range*, defined for spaces with a tensor product structure [32], one can introduce the numerical shadow restricted to the subset of separable (product) states or the set of maximally entangled states [12]. Analysing such restricted numerical shadows for operators of a composite dimension  $NM$  one may thus investigate the geometry of the selected set of separable (maximally entangled) quantum pure states. Such an approach is advocated in a forthcoming publication [33].

## Acknowledgments

It is a pleasure to thank G. Auburn and S. Weis for fruitful discussions. Work by J. Holbrook was supported in part by an NSERC of Canada research grant. Work by P. Gawron and J.A. Miszczak was supported by the Polish Ministry of Science and Higher Education under the grant number N519 442339, Z. Puchała was supported by Polish Ministry of Science and Higher Education under the project number IP 2010 033 470, while K. Życzkowski acknowledges support by the Polish Ministry of Science and Higher Education grant number N202 090239. Authors would like to thank S. Opozda for his help with the preparation of 3D models.

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